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A METHOD OF ANALYSIS OF PLATES REINFORCED WITH DISCRETE STIFFENERS

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Abstract—This paper attempts to demonstrate a new method for analyzing problems involving nonhomogeneous discretely reinforced plates. The paper shows that a uniform plate can be accurately synthesized by a structure made up entirely of simple beams and twisting elements. Using a second order approximation for the equations governing the bending elements it becomes possible to specify the distribution of some of the internal stresses. This additional flexibility makes it possible to independently specify bending and torsional stiffness as well as the distribution of the interaction moment. The model is shown to converge in the limit to the governing differential equation of a uniform plate. The equations for a discretely reinforced uniform plate are derived. The equations derived are applied to a uniformly loaded simply supported plate. The accuracy of the results for a 4×4 grid is superior to that obtained with finite difference procedure for the given internal stress distribution assumed.

NOTATION

EI	flexural rigidity (beam)
D	flexural rigidity plate
h	grid spacing
ν	Poisson's ratio
$Z1$	see Appendix
$Z2$	see Appendix
$Z3$	see Appendix
b	half width of beam reinforcement
q	load per unit area
a	plate width

Subscripts

m, n	location of applied vector, first subscript designates direction of application
mn	indicates differentiation with respect to m then n

INTRODUCTION

THE magnitude of the effort in the area of structural analysis of plates and shells has led to the development of a great many methods and techniques for their solution [1]. Accurate analyses in the simpler cases are dependent on a knowledge of the governing differential equations for the plate or shell. The analyst may then, through the use of various "smearing" techniques, extrapolate the use of these equations so that the results represent the states of complex plates or shells even when discretely reinforced.

The finite element method which at present represents the most powerful method for representing structures has undergone a fast and significant development within the last 10 yr. It first appeared in 1941 when Hrennikoff [2] approximated a flat rectangular plate

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by an assemblage of trusses composed of six pin jointed bar elements. With the advent of computers this method has been greatly extended by Hrennikoff [3], Clough [4], Yettram and Husain [5, 6], McCormick [7], Turner *et al.* [8], Lunder [9], Zienkiewicz [10], Argyris [11, 12].

Application of the finite element method is based upon prescribing variations of the field variables within a sufficiently small discrete region of the continuum under consideration. The functions of these prescribed field variables are assumed to be defined in terms of an arbitrary set of values of the field functions at a selected number of grid points. Then a variational principle is applied and the field function at those preselected grid points are subjected to arbitrary variations or virtual changes. These variational principles which are expressed in terms of volume integrals are then extended to include the entire discretized continuum. The displacements and their derivatives or the stresses or a combination of stresses, displacements and displacement derivatives may be considered as the arbitrary unknown field functions depending on the variational formulation (stiffness or flexibility method).

A unique minimizing solution is assumed to exist if the functionals are everywhere finite. Thus arbitrary field functions making up the functionals and their derivatives must be continuous across the boundaries of the element up to and including an order one less than that appearing in the functional. Therefore for finite values of the functional, one must have continuity of the transverse and in plane displacements as well as the first derivative of the transverse displacements across the boundaries of the discrete element. It is also necessary that the approximations be such as to include the possibility of a constant value of the functional throughout the element.

The equivalence between Galerkin's method and the finite element method has been established by Reddi [13]. In virtue of this equivalence, application of the finite element method need no longer require the variational principle, but may be applied directly.

Although the selection of different panel or discrete continuum element shapes and displacement modes lead to different capabilities, it is still useful to consider an example. A rectangular element with four discrete corner points, when the transverse deflection alone is considered, may have an acceptable set of displacement functions generated with only twelve degrees of freedom. The resultant set as commonly used will not satisfy inter-element slope continuity requirements. Nonetheless, this set will prove to be satisfactory under the relaxed necessary and sufficient convergence conditions as noted by Bozely *et al.* [14]. Examples of the use of displacement functions with only 12° of freedom are given by Zienkiewicz [10], Zudans [15], Melosh [16]. Improvements by Argyris [17] and Dawe [18] are accomplished by adding to the number of degrees of freedom thus allowing partial control of the magnitude of the slope discontinuity for general improvement in final results.

The finite element method as outlined above, does not provide for the satisfaction of internal equilibrium within the element (displacement method) only the overall equilibrium of the element is satisfied. Also, the concept of replacing distributed tractions by equivalent static loads may raise some questions as to the precise physical conditions to which the structure is being subjected.

The method developed below differs significantly from those methods discussed above yet in a broad sense may be considered a finite element method, distantly related to the Hrennikoff idea. The method as developed hypothesizes a set of elementary twisting and bending elements and their mode of interaction. The assemblage or panel is constructed and permitted to become small in order to demonstrate that in the limit the biharmonic plate equation can be obtained. A "second order" model is constructed using the elements of the original panel from which the final difference equations are obtained.

The panel developed has twelve degrees of freedom as did the finite element representation discussed above; however, in this panel deflection, slope and moment continuity are maintained between panels at the four corners of the element. The advantage over "shearing" methods in finite differences is demonstrated. The panel developed satisfies internal as well as external equilibrium requirements. Loads may be varied arbitrarily in the coordinate directions, even to the "exact" representation of a concentrated load anywhere on the plate. Internal stress distributions may also be controlled for greater accuracy in extreme cases.

This panel can be developed so as to "model" or simulate stiffness criteria obtained from additional stiffeners or experimental findings. Zudans treats this case for the finite element method [15]. This flexibility is possible because the torsional and bending stiffness are found to be decoupled in this development. The additional flexibility obtained is at the expense of a greater initial difficulty in the development of the appropriate finite difference equations for the new panel.

METHODOLOGY

Homogeneous plate

Let us determine the strength and stiffness of a plate which contains uniformly spaced discrete stiffeners. The stiffeners and plate are assumed to be of uniform thickness and dimension and to be made up of an homogeneous material. We consider as an element of the plate the exploded view shown in Fig. 1. The dashed lines in the figure make up the gridwork and also serve as a coordinate system.

It is hypothesized that the four rectangular elements, A, B, C, D are pure twisting elements and represent all the effective torsional stiffness of the area contained. The elements A, B, C and D are assumed to be acted upon by concentrated forces at the corners of the plate in accordance with the results obtained for a rectangular plate subjected to a pure torque using the Kirchoff conditions. For convenience, these torsional elements (actually subelements) are all assumed to be identical. It is further postulated that the bending stiffness of the element is determined by the vertical member or rib acting in

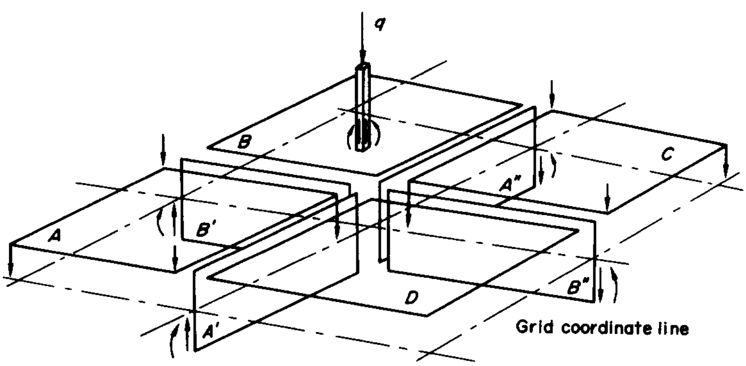


FIG. 1. Schematic view showing typical element.

conjunction with that portion of the plate which lies half a grid space on either side of the vertical rib. The spacing of the coordinate lines is chosen so that the "effective width criteria" is not exceeded anywhere in the plate. Thus the beam cross section is determined to be that of a T-section (flange plus the vertical rib). The bending stiffness is computed accordingly. It should be noted that the flange width is fixed by the original choice of coordinate lines.

When the vertical ribs are welded or molded together, the additional stiffness obtained from the "interaction moment" must be taken into account. The term "interaction moment" refers to that moment generated in the $B'B''$ by an applied moment, M, shown in Fig. 2 on the beam $A'A''$.

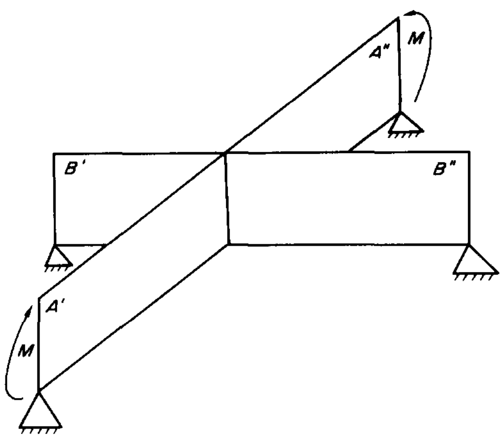


FIG. 2. Interaction moment.

Let us use the grid coordinate system shown in Fig. 3.

It is easy to show that if we consider only beam bending and include the effects of the transverse moment, we can write

$$(M_{m,n} - vM_{n,m})/EI = (y_{m+1,n} - 2y_{m,n} + y_{n-1,m})/h^2 = A. \quad (1)$$

Similarly

$$(M_{n,m} - vM_{m,n})/EI = (y_{n+1,m} - 2y_{n,m} + y_{n-1,m})/h^2 \quad (2)$$

where the right hand side of the equation represents the curvature and the left, the moment per unit length less the Poisson ratio times the interaction moment. The result is the effective moment producing curvature at the point (n, m). Solving for $M_{m,n}$ and $M_{n,m}$, we find

$$M_{m,n} = EI/(h^2(1-v^2))(A+vB) \quad (3)$$

$$M_{n,m} = EI/(h^2(1-v^2))(B+vA) \quad (4)$$

where the order of the subscripts on vector or directional quantities such as moment, slope, shear, etc. indicates direction. Thus the subscript designation (m, n) signifies the point (m, n) and the first subscript, m, signifies the vector direction. Thus $M_{m,n}$ is the moment

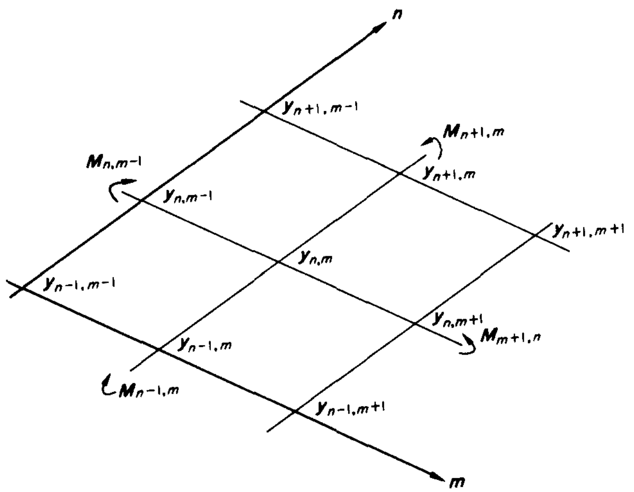


FIG. 3. Coordinate system, vector designation scheme.

located at the point (m, n). The moment is applied in the m direction. If equations (3) and (4) are differentiated twice with respect to the m and n direction respectively, we find

$$M_{m+1,n}-2M_{m,n}+M_{m-1,n} = D(A_{mm}+\nu B_{mm})$$

$$M_{n+1,m}-2M_{n,m}+M_{n-1,m} = D(B_{nn}+\nu A_{nn}).$$

Summing we obtain,

$$\bar{q} = D(A_{mm}+B_{nn}+\nu(A_{nn}+B_{mm}).$$

Let us consider the action of the torsional elements depicted as plates which lie interior to the grid lines and make no contribution to bending strength. They are assumed to behave as if they are attached only at their corners to the grid coordinate points; i.e. where the beam elements cross. The reacting load at these points is assumed to be a concentrated load. This may be justified either from elementary considerations or plate theory using the Kirchoff conditions. The force exerted by the torsional elements A, B, C, D, may be written as

$$\begin{aligned}\tfrac{1}{4}\{&-(y_{n-1,m}+y_{n,m+1}-y_{n,m}-y_{n-1,m+1})+(y_{n+1,m+1}+y_{n,m}-y_{n,m+1}-y_{n+1,m})\\&+(y_{n-1,m-1}+y_{n,m}-y_{n,m-1}-y_{n-1,m})-(y_{n,m-1}+y_{n+1,m}-y_{n,m}-y_{n+1,m-1})\}\\&\times 8D(1-\nu^2)/h^2 = R\end{aligned} \tag{5}$$

where R is the total load exerted by all the torsional elements on the beams at the grid intersection (m, n). This is not inconsistent with the finite difference representation.

The sum of the load acting at the grid from all elements at the coordinate (m, n) is given as

$$\bar{q}+R = qh^2. \tag{6}$$

The result in the limit is the finite difference representation of the biharmonic equation governing plate deflection.

$$\nabla^4 y = q/D. \tag{7}$$

Discretely reinforced plates

Although the representation obtained above is for a uniform plate and its behavior in the limit is encouraging, it does not give additional insight into plate behavior nor aid in solving more general problems. Further, though the model precisely measures stresses and deflections where the rib flange ratio is one, it is of more interest to determine it in the case where the ratio is different from one. In order to find this solution, it is necessary to eliminate dependence on the finite difference formulation of the Bernoulli beam bending equation since in that formulation the moment distribution is described by straight line segments between points at which deflection is measured. This distribution will not permit variation of the rib flange ratio. In order to achieve the required generality the finite difference form of the Bernoulli equation is replaced by the equivalent three moment equation which provides more accurate moment estimates between grid points.

In order to derive an equivalent load deflection relationship using the three moment equation, assumptions must be made regarding internal stress variations due to the individual subelement interactions. The only precedent in the literature to guide the decision is the finite difference formulation which assumes the interaction may be represented as concentrated loads which is a conservative estimate. The distribution assumed here is shown in Fig. 4 where the interaction loads are considered to be uniformly distributed over half the grid distance on either side of a grid coordinate point on the beam element. The interaction loads are the loads generated at the corners of the twisting element and also the internal load generated between the crossed bending elements. The external loads $q_{m,n}$ are considered to be acting uniformly from grid point to grid point on the beam element shown in Fig. 4.

The relationship of the loading is schematically represented in Fig. 5.

Thus we may establish the relationship that

$$\bar{q}_{n,m} = \bar{\bar{q}}_{m,n} + \tilde{q}_{n,m} \tag{8}$$

where $\bar{q}_{n,m}$ is the entire load from the torsional elements acting over four quadrants at the intersection (n, m). Thus as before the resultant load from the torsional elements may be expressed as

$$R/h = \bar{q}_{n,m}. \tag{9}$$

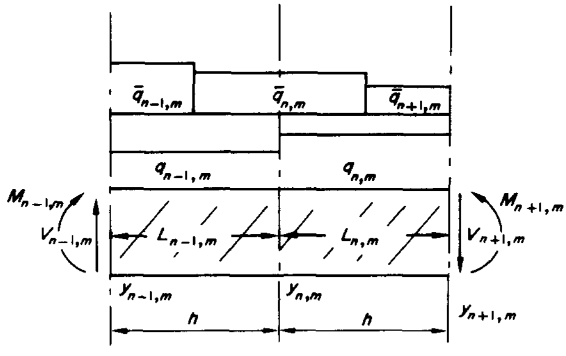


FIG. 4. Beam element loading.

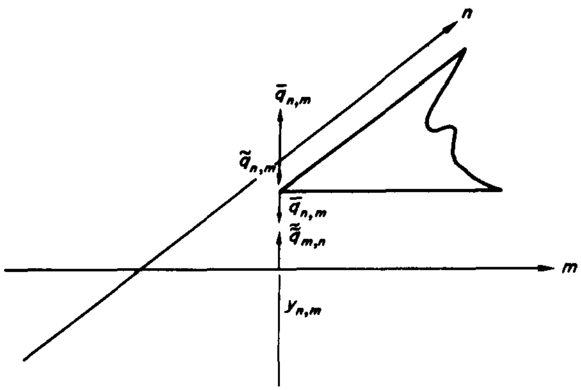


FIG. 5. Schematic view of interaction loads.

Using the equilibrium equations on the element in Fig. 4, we find that

$$M_{n+1,m}-2M_{n,m}+M_{n-1,m} = -\tilde{q}_{n+1,m}h^2/8-3/4\tilde{q}_{n,m}h^2-q_{n,m}h^2/2-q_{n-1,m}h^2/2-\tilde{q}_{n-1,m}h^2/8 \quad (10)$$

The application of the three moment equation to this element results in the following equation

$$M_{n-1,m}+4M_{n,m}+M_{n+1,m} = 6EI/h^2(y_{n+1,m}-2y_{n,m}+y_{n-1,m})-h^2/4(q_{n-1,m}-q_{n,m}) -h^2/128(50\tilde{q}_{n,m}-7\tilde{q}_{n-1,m}-7\tilde{q}_{n+1,m}). \quad (11)$$

Adding equations (5) and (6) together to eliminate loads with subscripts other than (n, m) it is found that two equations result from the beams crossing in the n, and m directions respectively whose centers lie at (m, n). They are

$$M_{n-1,m}+46M_{n,m}+M_{n+1,m} = 48EI/h^2(y_{n+1,m}-2y_{n,m}+y_{n-1,m}) +h^2[3\bar{q}_{n,m}+1{\cdot}5(q_{n-1,m}+q_{n,m})]. \quad (12)$$

A similar one may be obtained through subscript interchange except that $q_{n,m}$ is replaced with $q_{m,n}$. Summing these two equations and using equation (8) then substituting (9), we finally obtain

$$M_{m-1,n}+M_{m+1,n}+46M_{m,n}+M_{n-1,m}+M_{n+1,m}+46M_{n,m} = z^1. \quad (13)$$

We may obtain two more independent relationships which are not dependent on the internal loads by summing equation (11) and its counterpart in the m direction as shown below

$$\begin{aligned}
M_{n-1,m-1}+4M_{n,m-1}+M_{n+1,m-1} &= \ldots -\tilde{q}_{n+1,m-1}7h^2/128\\
M_{n-1,m}+4M_{n,m}+M_{n+1,m} &= \ldots -\tilde{q}_{n+1,m}7h^2/128\\
M_{n-1,m+1}+4M_{n,m+1}+M_{n+1,m+1} &= \ldots -\tilde{q}_{n+1,m+1}7h^2/128\\
M_{m-1,n+1}+4M_{m,n+1}+M_{m+1,n} &= \ldots -\tilde{\tilde{q}}_{m+1,n+1}7h^2/128\\
M_{m-1,n}+4M_{m,n}+M_{m+1,n} &= \ldots -\tilde{\tilde{q}}_{m+1,n}7h^2/128\\
M_{m-1,n-1}+4M_{m,n-1}+M_{m+1,n-1} &= \ldots -\tilde{\tilde{q}}_{m+1,n-1}7h^2/128.
\end{aligned} \quad (14)$$

Premultiplying the second and fifth equation of the set (14) by 50/7 and summing the entire set of equations will eliminate the internal loads and obtain equation (15) as follows

$$\begin{aligned} &M_{n-1,m-1}+4M_{n,m-1}+M_{n+1,m-1}+50/7(M_{n-1,m}+4M_{n,m}+M_{n+1,m})+M_{n-1,m+1} \\ &\quad +4M_{n,m+1}+M_{n+1,m+1}+M_{m-1,n-1}+4M_{m,n-1}+M_{m+1,n-1}+50/7(M_{m-1,n} \\ &\quad +4M_{m,n}+M_{m+1,n})+M_{m-1,n+1}+4M_{m,n+1}+M_{m+1,n+1} = Z2. \end{aligned} \tag{15}$$

Proceeding in a similar manner using equation (10), we find respectively the following result

$$\begin{aligned} &M_{n+1,m+1}-2M_{n,m+1}+M_{n-1,m+1}+6(M_{n+1,m}-2M_{n,m}+M_{n-1,m})+M_{n+1,m-1} \\ &\quad -2M_{n,m-1}+M_{n-1,m-1}+M_{m+1,n+1}-2M_{m,n+1}+M_{m-1,n+1}+6(M_{m+1,n} \\ &\quad -2M_{m,n}+M_{m-1,n})+M_{m+1,n-1}-2M_{m,n-1}+M_{m-1,n-1} = Z3. \end{aligned} \tag{16}$$

Equations (13), (15), (16) define the relationship needed to satisfy the requirements of plate compatibility conditions.

Let us now consider the effect of a concentrated transverse couple on the terms in the three moment equation. If a concentrated transverse couple M is applied to a section of a beam, the effect is precisely the same as applying a couplet† to the beam at that section acting along the beam of magnitude νM. The effect of a distribution of these couplets may be found through an integration of the resulting M/EI diagram which is shown as

$$\begin{aligned} &M_{n+1,m}L_{n,m}+2M_{n,m}(L_{n-1,m}+L_{n,m})+M_{n-1,m}(L_{n-1,m}) = 6\nu\left\{\frac{1}{L_{n,m}}\int_0^{L_{n,m}}\ldots F_{n,m}(M_{m,n+1};\right. \\ &\quad \left.\times M_{m,n};\xi)\xi\,\mathrm{d}\xi+1/L_{n-1,m}\int_0^{L_{n-1,m}}F_{n-1,m}(M_{m,n+1};M_{m,n};\xi)\xi\,\mathrm{d}\xi\right\} \\ &\quad +6EI(y_{n-1,m}/L_{n-1,m}+y_{n+1,m}/L_{n,m}). \end{aligned} \tag{17}$$

Where the function $F_{n,m}$ are arbitrary internal distributions of interaction moments chosen by the analyst to satisfy most nearly reinforcement configurations in a given design. If the case is considered where the distribution is assumed to vary linearly from grid point to grid point (as is done in the case of a uniform plate; rib flange ratio, one), $F_{n,m}$ taken on the form

$$F_{n,m} = (M_{m,n}-M_{m+1,n})/L_{n,m}\xi+M_{m+1,n}. \tag{18}$$

If the rib width of the reinforcement is less than grid spacing (i.e. rib flange ratio less than one) and if the interaction moment is considered constant over the thickness of the

† Two equal and opposite couples acting an infinitesimal distance apart.

web, the function $F_{n,m}$ takes on the form

$$F_{n,m} = \begin{cases} M_{m,n} & L_{m,n}-b < \xi < L_{m,n} \\ 0 & b < \xi\, L_{m,n}-b \\ M_{m+1,n} & 0 < \xi < b \end{cases}$$

where it is assumed the intermediate plate interaction moment† is negligible and the rib width is $2b$ everywhere.

In order to consider the case where the interaction variation is linear as shown in Fig. 6, all that is required is to replace $M_{n,m}$ as

$$M_{n,m} \doteq M_{n,m} - \nu M_{m,n} \tag{19}$$

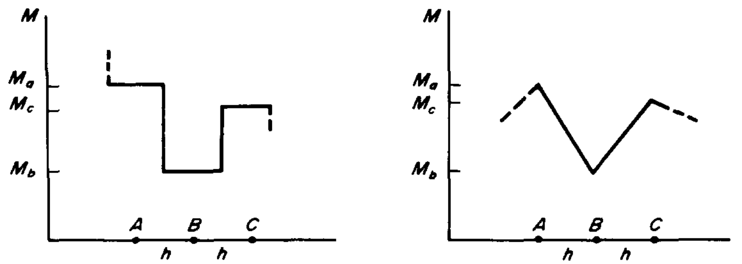


FIG. 6. Assumed variations of interaction moment.

in each application of the three moment equation in the above derivation, for which it is interesting to note that the same substitution is used to obtain equations (1) and (2). A very similar relationship may be found for the case where the moment is assumed to vary in a stairstep pattern shown in Fig. 6(a).

Making the substitution from equation (19) into equations (13), (15), 16) we obtain the final form of the equations which determine the stress and deflections in a uniform plate (rib flange ratio one) with a linear variation in the interaction moment from grid point to grid point.

$$\begin{aligned} &M_{m-1,n}+M_{m+1,n}+46M_{m,n}+M_{n-1,m}+M_{n+1,m}+46M_{n,m}-8\nu(M_{n,m-1}+4M_{n,m} \\ &\quad +M_{n,m+1}+M_{m,n-1}+4M_{m,n}+M_{m,n+1}) = Z1 \end{aligned} \tag{20}$$

$$\begin{aligned} &M_{n-1,m-1}+4M_{n,m-1}+M_{n+1,m-1}+50/7(M_{n-1,m}+4M_{n,m}+M_{n+1,m})+M_{n-1,m+1} \\ &\quad +4M_{n,m+1}+M_{n+1,m+1}+M_{m-1,n-1}+4M_{m,n-1}+M_{m+1,n-1}+50/7(M_{m-1,n}+4M_{m,n} \\ &\quad +M_{m+1,n})+M_{m-1,n+1}+4M_{m,n+1}+M_{m+1,n+1}-\nu(M_{m-1,n+1}+4M_{m-1,n}+M_{m-1,n-1} \\ &\quad +50/7(M_{m,n+1}+4M_{m,n}+M_{m,n-1})+M_{m+1,n+1}+4M_{m+1,n}+M_{m+1,n-1}+M_{n-1,m-1} \\ &\quad +M_{n-1,m}+M_{n-1,m+1}+50/7(M_{n,m-1}+4M_{n,m}+M_{n,m+1})+M_{n+1,m-1}+4M_{n+1,m} \\ &\quad +M_{n+1,m+1}) = Z2 \end{aligned} \tag{21}$$

† In the region where $b < \xi < L_{m,n}-b$ the function $F_{n,m}$ must be determined for the interaction of plate with reinforcement. Estimation of this factor must be left to the individual analyst or the results of a series of physical tests. For convenience it is here assumed zero.

$$M_{n+1,m+1}-2M_{n,m+1}+M_{n-1,m+1}+6(M_{n+1,m}-2M_{n,m}+M_{n-1,m})+M_{n+1,m-1}$$
$$-2M_{n,m-1}+M_{n-1,m-1}+M_{m+1,n+1}-2M_{m,n+1}+M_{m-1,n+1}+6(M_{m+1,n}-2M_{m,n} \quad (22)$$
$$+M_{m-1,n})+M_{m+1,n-1}-2M_{m,n-1}+M_{m-1,n-1} = Z3.$$

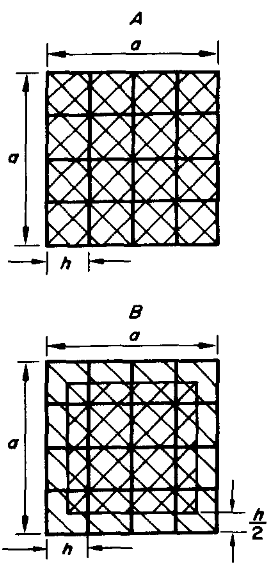
In order to determine the deflections and stresses for a plate discretely reinforced with ribs of width $2b$, where $h < 2b$, substitution of the result of the integration from equations (17) and (18) which is given as

$$vM_{m,n} \doteq 3vb(2M_{m,n}(2h-b))/2h$$
$$vM_{m,n+1} \doteq 3vb^2M_{m,n+1}/h$$
$$vM_{m,n-1} \doteq 3vb^2M_{m,n-1}/h$$

into equations (20) and (21) obtains the set of equations governing the stresses and deflections in an arbitrarily loaded uniform plate, discretely reinforced. The stiffeners of which are of width $2b$ and the grid spacing is h.

RESULTS

In order to demonstrate the accuracy obtainable using this method and the effect of parameter variations the final equations were applied to the case of a uniformly loaded simply supported square plate. Two parameter variations were considered. In one case the interaction moment was permitted to vary in a stairstep pattern as shown in Fig. 4(a), and in the other a linear variation as shown in Fig. 4(b). The load distribution pattern was also varied slightly while the interaction moment was permitted to vary linearly. Figure 7



Condition – plate A, interaction moments are controlled to vary linearly from values established at grid point

$y_{max} = 0{\cdot}00413\ qa^4/D$
$M_{max} = 0{\cdot}0479\ qa^2$

Condition – plate A, interaction moments are controlled to vary in stair step pattern (constant over grid intersection to half interval points)

$y_{max} = 0{\cdot}004075\ qa^4/D$
$M_{max} = 0{\cdot}04825\ qa^2$

Condition – plate B, interaction moments are controlled to vary linearly from value established at grid point

$y_{max} = 0{\cdot}00440\ qa^4/D$
$M_{max} = 0{\cdot}05055\ qa^2$

FIG. 7.

shows the results obtained in each case. For the reader's convenience, the finite difference solution is given as

$$y_{max} = 0{\cdot}00403qa^4/D; \qquad M_{max} = 0{\cdot}0457qa^2$$

and the exact solution as

$$y_{max} = 0{\cdot}00406qa^4/D; \qquad M_{max} = 0{\cdot}0479qa^2.$$

The error found using stairstep interaction moment variations are less than 1 per cent for both the deflection and the moment using a 4 × 4 grid.

The grid size used in all calculations shown is 4×4. The loading used in each case is shown by crosshatching—the singly crosshatched area carrying twice the load of the doubly crosshatched area. The loading it should be noted is precisely as shown.

CONCLUSIONS

The method described above for the construction and analyses of a plate element from an ensemble of elementary pure twisting and bending elements has been shown to be capable of providing an accurate means for calculating stresses and deflections in plates. The extension of this method to handle nonhomogeneous or general boundary value problems for discretely reinforced plates may be accomplished in a relatively straightforward manner from the derivation presented above. The resulting program should be capable of being easily modified to incorporate design changes as only three entries become significant. They are the incremental bending and torsional stiffness and the modification to the interaction moment distribution.

It should be noted that the size of the matrices do become large as the number of grid points increases because three quantities must be determined at each grid point. However, the example seems to indicate that a coarser grid may be tolerated without significant loss in accuracy. This in effect may permit the generality of the approach to be conserved with no increase in the amount of computer storage necessary over finite element methods for a given precision in the results.

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APPENDIX

The quantities $Z1, Z2, Z3$ are defined as follows:

$$\begin{aligned} Z^1 = {} & 48EI/h^2(y_{n+1,m}+y_{m+1,n}-4y_{m,n}+y_{n-1,m}+y_{m-1,n})+3/2h^2(q_{n,m}+q_{n-1,m} \\ & +q_{m,n}+q_{m-1,n})+6D(1-\nu)/h(y_{n-1,m+1}-2y_{n,m+1}+y_{n+1,m+1}-2(y_{n+1,m}-2y_{n,m} \\ & +y_{n-1,m})+y_{n+1,m-1}-2y_{n,m-1}+y_{n-1,m-1}), \end{aligned}$$

$$\begin{aligned} Z2 = {} & 12EI/h^2(y_{n-1,m-1}+18/7y_{n,m-1}+y_{n+1,m-1})+24EI/(7h^2)\{y_{n-1,m}-50y_{n,m} \\ & +y_{n+1,m}\}+12EI/h^2[y_{n-1,m+1}+18/7y_{n,m+1}+y_{n+1,m+1}]-h^2/4(50/7(q_{n-1,m}+q_{n,m}) \\ & +q_{n-1,m+1}+q_{n,m+1}+q_{n-1,m-1}+q_{n,m-1}+50/7q_{m-1,n}+50/7q_{m,n}+q_{m-1,n+1} \\ & +q_{m,n+1}+q_{m-1,n-1}+q_{m,n-1})+2D(1-\nu)/h\{7/128y_{n-2,m+2}+9/32y_{n-1,m+2} \\ & -43/64y_{n,m+2}+9/32y_{n+1,m+2}+7/128y_{n+2,m+2}+9/32y_{n-2,m+1}+81/56y_{n-1,m+1} \\ & -183/56y_{n,m+1}+81/56y_{n+1,m+1}+9/32y_{n+2,m+1}-43/64y_{n-2,m}-183/56y_{n-1,m} \\ & -1849/224y_{n,m}-183/56y_{n+1,m}-43/64y_{n+2,m}+9/32y_{n-2,m-1}+81/56y_{n-1,m-1} \\ & -183/56y_{n,m-1}+81/56y_{n+1,m-1}+9/32y_{n+2,m-1}+7/128y_{n-2,m-2} \\ & +9/32y_{n-1,m-2}-43/64y_{n,m-2}+9/32y_{n+1,m-2}+7/128y_{n+2,m-2}\}, \end{aligned}$$

$$\begin{aligned} Z3 = {} & -1/2\{q_{n,m+1}+q_{n-1,m+1}+6q_{n,m}+6q_{n-1,m}+q_{n,m-1}+q_{n-1,m-1}+q_{m,n+1} \\ & +q_{m-1,n+1}+6q_{m,n}+6q_{m-1,n}+q_{m,n-1}+q_{m-1,n-1}\}-D(1-\nu)/h^3[+1/4(y_{n-2,m+2} \\ & +4y_{n-1,m+2}-10y_{n,m+2}+4y_{n+1,m+2}+y_{n+2,m+2})+(y_{n-2,m+1}+4y_{n-1,m+1} \\ & -10y_{n,m+1}+4y_{n+1,m+1}+y_{n+2,m+1})-5/2(y_{n+2,m}+4y_{n-1,m-1}-10y_{n,m-1} \\ & +4y_{n+1,m}+y_{n+2,m})+(y_{n-2,m-1}+4y_{n-1,m-1}-10y_{n,m-1}+4y_{n+1,m-1}+y_{n+2,m-1}) \\ & +1/4(y_{n-2,m-2}+4y_{n-1,m-2}-10y_{n,m-2}+4y_{n+1,m-2}+y_{n+2,m-2})]. \end{aligned}$$

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Абстракт—Работа предпринимается представить новый метод анализа задач, касающихя неоднородных, усиленных дискретно пластинок. Работа указывает, что однородная пластинка может быть точно обобщена с помощью конструкции, сложенной полностью из простых балок и элементов подверженных кручению. Пользуясь аппроксимацией второго порядка для решения уравнений, описывающих поведения изгибаемых элементов, оказывается возможным определить распределение некоторых внутренних напряжений. Благодаря этому методу можно определить, независимо, жесткости изгиба и кручения, а также распределение взаимодействующего момента. Предлагаемая модель указывает, что в предельном переходе она стремится к дифференциальному определяющему уравнению однородной пластинки. Выводятся уравнения для дискретно усиленной однородной пластинки. Эти уравнения применяются к расчету равномерно нагруженной свободно опертой пластинки. Точность результатов для решетки оказывается высшая по сравнению с результатом полученным методом конечных разностей для заданного распределения внутренних напряжений.